Title: Remarks on generalized solutions of some ordinary nonlinear differential equations of second order in the Columbeau algebra

Author: Jan Ligęza

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Prace Naukowe Uniwersytetu Śląskiego nr 1564

REMARKS ON GENERALIZED SOLUTIONS OF SOME ORDINARY NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN THE COLUMBEAU ALGEBRA

JAN LIGĘZA

Abstract. In this article some equations of second order are considered, whose nonlinearity satisfies a global Lipschitz condition. It is shown that the equations with additional conditions admit unique global solutions in the Colombeau algebra $\mathcal{G}(\mathbb{R}^1)$.

1. Introduction

We consider the following problems

$$x''(t) + p(t)f_1(t, x(t), x'(t)) + q(t)f_2(t, x(t), x'(t)) = r(t), \tag{1.0}$$

$$x(a) = d_1, \qquad x'(a) = d_2, \qquad a \in \mathbb{R}^1, \ \ d_1, d_2 \in \overline{\mathbb{R}}, \tag{1.1}$$

$$x(a) = r_1, \qquad x(b) = r_2, \qquad a, b \in \mathbb{R}^1, \ \ a < b, \ \ r_1, r_2 \in \overline{\mathbb{R}}, \tag{1.2}$$

where p, q and r are elements of the Colombeau algebra $\mathcal{G}(\mathbb{R}^1)$; $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}^1$ are smooth functions ($f_1, f_2 \in C^\infty(\mathbb{R}^3)$); d_1, d_2, r_1, r_2 are known elements of the Columbeau algebra $\overline{\mathbb{R}}$ of generalized real numbers; $x(a), x'(a), x(b)$ are understood as the value of the generalized functions x and x' at the points a and b respectively (see [2]). Elements p, q, r, f_1 and f_2 are given. The

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derivative, the sum, the equality and the superposition are meant in the Colombeau algebra sense (see [2]).

We prove theorems on existence and uniqueness of solutions of the problems (1.0) – (1.1) and (1.0); (1.2). In the paper [2] some differential equations with coeffeciEnts from the Colombeau algebra were examined. Certain problems for the quantum theory lead to such equations. Our results generalize some results given in [11] and [12].

2. Notation

Let $\mathcal{D}(\mathbb{R}^1)$ be the set of all C^∞ functions $\mathbb{R}^1 \to \mathbb{R}^1$ with compact support. For $q = 1, 2, \dots$ we denote by $\mathcal{A}q$ the set of all functions $\phi \in \mathcal{D}(\mathbb{R}^1)$ such that relations

$$(2.0) \qquad \int_{-\infty}^{\infty} \phi(t)dt = 1, \qquad \int_{-\infty}^{\infty} t^k \phi(t)dt = 0, \qquad 1 \le k \le q$$

hold.

Next, $\mathcal{E}[\mathbb{R}^1]$ is the set of all functions $R: \ \mathcal{A}_1 \times \mathbb{R}^1 \to \mathbb{R}^1$ such that $R(\phi, t) \in C^\infty$ for every fixed $\phi \in \mathcal{A}_1$.

If $R \in \mathcal{E}[\mathbb{R}^1]$, then $D_k R(\phi, t)$ for any fixed ϕ denotes a differential operator in t (i.e. $D_k R(\phi, t) = \frac{d^k}{dt^k}(R(\phi, t))$).

For given $\phi \in \mathcal{D}(\mathbb{R}^1)$ and $\varepsilon > 0$ we define ϕ_ε by

$$(2.1) \qquad \phi_\varepsilon(t) = \frac{1}{\varepsilon}\phi\left(\frac{t}{\varepsilon}\right).$$

An element R of $\mathcal{E}[\mathbb{R}^1]$ is moderate if for every compact set K of $\mathbb{R}^1$ and every differential operator D_k there is $N \in \mathbb{N}$ such that the following condition holds: for every $\phi \in \mathcal{A}_N$ there are $\varepsilon > 0$, $\eta > 0$ such that

$$(2.2) \qquad \sup_{t \in K} |D_k R(\phi_\varepsilon, t)| \le c\varepsilon^{-N} \qquad \text{if} \quad 0 < \varepsilon < \eta.$$

We denote by $\mathcal{E}_M[\mathbb{R}^1]$ the set of all moderate elements of $\mathcal{E}[\mathbb{R}^1]$.

By Γ we denote the set of all increasing functions α from $\mathbb{N}$ into $\mathbb{R}^1_+$ such that $\alpha(q)$ tends to ∞ if $q \to \infty$.

We define an ideal $\mathcal{N}[\mathbb{R}^1]$ in $\mathcal{E}_M[\mathbb{R}^1]$ as follows: $R \in \mathcal{N}[\mathbb{R}^1]$ if for every compact set K of $\mathbb{R}^1$ and every differential operator D_K there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that the following condition holds: for every $q \ge N$ and $\phi \in \mathcal{A}_q$ there are $c > 0$ and $\eta > 0$ such that

$$(2.3) \qquad \sup_{t \in K} |D_k R(\phi_\varepsilon, t)| \le c\varepsilon^{\alpha(q) - N} \qquad \text{if} \quad 0 < \varepsilon < \eta.$$

The algebra $\mathcal{G}(\mathbb{R}^1)$ (the Colombeau algebra) is defined as quotient algebra of $\mathcal{E}_M[\mathbb{R}^1]$ with respect to $\mathcal{N}[\mathbb{R}^1]$ (see [2]).

We denote by $\mathcal{E}_0$ the set of all the functions from $\mathcal{A}_1$ into $\mathbb{R}^1$. Next, we denote by $\mathcal{E}_M$ the set of all the so–called moderate elements of $\mathcal{E}_0$ defined by

(2.4) $\mathcal{E}_M = \{R \in \mathcal{E}_0$: there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are $c > 0,\ \eta > 0$ such that $|R(\phi_\varepsilon)| \leq c\varepsilon^{-N}$ if $0 < \varepsilon < \eta\}$.

Further, we define an ideal $\mathcal{T}$ of $\mathcal{E}_M$ by

(2.5) $\mathcal{T} = \{R \in \mathcal{E}_0$: there are $N \in \mathbb{N}$ and $\alpha \in \Gamma$ such that for every $q \geq N$ and

$\phi \in \mathcal{A}q$ there are $c > 0,\ \eta > 0$ such that $|R(\phi_\varepsilon)| \leq c\varepsilon^{\alpha(q)-N}$ if $0 < \varepsilon < \eta\}$.

We define an algebra $\overline{\mathbb{R}}$ by setting

$$\overline{\mathbb{R}} = \frac{\mathcal{E}_M}{\mathcal{T}} \qquad \text{(see [2]).}$$

If $R \in \mathcal{E}_M[\mathbb{R}^1]$ is a representative of $G \in \mathcal{G}(\mathbb{R}^1)$, then for a fixed t the map $Y : \phi \to R(\phi, t) \in \mathbb{R}^1$ is defined on $\mathcal{A}_1$ and $Y \in \mathcal{E}_M$. The class of Y in $\overline{\mathbb{R}}^1$ depends only on G and t. This class is denoted by $G(t)$ and is called the value of the generalized function G at the point t (see [2]).

We say that a smooth function $f : \mathbb{R}^3 \to \mathbb{R}^1$ is polynomially bounded uniformly for t if for every compact interval K of $\mathbb{R}^1$ there are constants $c(K) > 0$ and $r \in \mathbb{N}$ such that

$$|f(t, u, v)| \leq c(K)(1 + |u| + |v|)^r \tag{2.6}$$

for all $u, v \in \mathbb{R}^1$ and $t \in K$.

We denote by $O_M(K, \mathbb{R}^2)$ the set of all the smooth functions $f : \mathbb{R}^3 \to \mathbb{R}^1$ which have the property that f and its partial derivatives are polynomially bounded uniformly for t.

If $f \in O_M(K, \mathbb{R}^2)$ and if $R_1, R_2 \in \mathcal{E}_M[\mathbb{R}^1]$, then $f(t, R_1, R_2) \in \mathcal{E}_M[\mathbb{R}^1]$ (see [2] p.29). If $f \in O_M(K, \mathbb{R}^2)$; $G_1, G_2 \in \mathcal{G}(\mathbb{R}^1)$, then an element of $\mathcal{G}(\mathbb{R}^1)$ denoted by $f(t, G_1, G_2)$ is defined as class of the functions $f(t, R_1, R_2)$, where $R_1, R_2 \in \mathcal{E}_M[\mathbb{R}^1]$ are representatives of G_1 and G_2 respectively.

We say that $x \in \mathcal{G}(\mathbb{R}^1)$ is a solution of the equation (1.0) if x satisfies the equation (1.0) identical in $\mathcal{G}(\mathbb{R}^1)$.

Throughout the paper K denotes a compact set in $\mathbb{R}^1$. We denote by $R_p(\phi, t), R_{x_0}(\phi),\ R_{x(t_0)}(\phi)$ representatives of elements p, x_0 and $x(t_0)$, respectively.

We put

$$\|x\|^1{}_{[a,b]} = \max_{t \in [a,b]} |x(t)| + \max_{t \in [a,b]} |x'(t)|, \qquad \text{if} \quad x \in C^1[a, b]$$

and

$$\|x\|_{[a,b]} = \max|x(t)|, \qquad \text{if} \quad x \in C_{[a,b]}.$$

The definition of generalized functions on an open interval $(A, B) \subset \mathbb{R}^1$ is almost the same as definition in the whole $\mathbb{R}^1$ (see [2]). In this paper we shall prove theorems on generalized solutions of nonlinear differential equations on $\mathbb{R}^1$. It is not difficult to observe that theorems proved are also true in the case when generalized functions p, q, r are considered on an interval (A, B) and $f_i : (A, B) \times \mathbb{R}^2 \to \mathbb{R}^1$, where $-\infty < A < a < b < B < \infty$.

3. The main results

First, we shall introduce a hypothesis H:

Hypothesis H

(3.0) $p, q, r \in \mathcal{G}(\mathbb{R}^1)$,

(3.1) the elements $p, q \in \mathcal{G}(\mathbb{R}^1)$ admit representatives $R_p(\phi, t)$ and $R_q(\phi, t)$ with the following properties: for every K there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are constants $c > 0$ and $\eta > 0$ such that

$$\sup_{t,t_0 \in K} \left| \int_{t_0}^{t} |R_p(\phi_\varepsilon, s)| ds \right| \leq c, \qquad \sup_{t,t_0 \in K} \left| \int_{t_0}^{t} |R_q(\phi_\varepsilon, s)| ds \right| \leq c$$

if $0 < \varepsilon < \eta$,

(3.2) $f_1, f_2 \in \mathcal{O}_M(K, \mathbb{R}^2)$,

(3.3) $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}^1$ are smooth functions such that for every $K \subset \mathbb{R}^1$ there are constants $M_{ij}(K) \geq 0$ such that

$$\left|\frac{\partial f_i}{\partial u_j}(t, u_1, u_2)\right| \leq M_{ij}(K) \text{ for } t \in K, \quad u_1, u_2 \in \mathbb{R}^1 \text{ and } i, j = 1, 2;$$

(3.4) the element $p \in \mathcal{G}(\mathbb{R}^1)$ admits a representative $R_p(\phi, t)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma > 0$ such that

$$I_1(p, \phi_\varepsilon) = M_{11} \int_a^b |R_p(\phi_\varepsilon, t)| dt \leq \frac{4}{b-a} - \gamma$$

$$\text{if} \quad 0 < \varepsilon < \varepsilon_0 \qquad (M_{11} = M_{11}([a, b])),$$

(3.5) the elements $p, q \in \mathcal{G}(\mathbb{R}^1)$ admit representatives $R_p(\phi, t)$ and $R_q(\phi, t)$ with the following property: there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ there are constants $\varepsilon_0 > 0$ and $\gamma > 0$ such that

$$\begin{aligned} I_2(p, q, \phi_\varepsilon) =& (M_{11} + M_{12}) \int_a^b |R_p(\phi_\varepsilon, t)| dt + (M_{21} + M_{22}) \int_a^b |R_q(\phi_\varepsilon, t)| dt \\ \leq& \frac{4}{b-a+4} - \gamma, \text{ if } \quad 0 < \varepsilon < \varepsilon_0 \qquad (M_{ij} = M_{ij}([a, b])). \end{aligned}$$

Now we shall give theorems on existence and uniqueness of the solution of the problems (1.0), (1.1) and (1.0), (1.2).

THEOREM 3.1. *We assume that the conditions (3.0)–(3.3) hold. Then the problem (1.0), (1.1) has exactly one solution x in $\mathcal{G}(\mathbb{R}^1)$.*

REMARK 3.1. Let δ denotes the generalized function (the Dirac's generalized delta function) which admits as the representative the functions $R_\delta(\phi, t) = \phi(-t)$, where $\phi \in \mathcal{A}_1$. Then δ has the property (3.1) (see [11]).

REMARK 3.2. It is not difficult to verify that the problem

$$x''(t) = 2\delta'(t)\delta(t)x'(t) \tag{3.6}$$

$$x(-1) = 0, \qquad x'(-1) = 1 \tag{3.7}$$

has not any solution in $\mathcal{G}(\mathbb{R}^1)$ (see [11]).

REMARK 3.3. Let $R_1(\phi, t) = \exp(\phi(-t))$, where $\phi \in \mathcal{A}_1$. Then $R_1(\phi, t) \notin \mathcal{E}_M[\mathbb{R}^1]$ (see [2], p.11). Nowe we define $R_2(\phi, t) = \sin(\phi(-t))$. We have $R_2(\phi, t) \in \mathcal{E}_M[\mathbb{R}^1]$.

THEOREM 3.2. *We assume the conditions (3.0)–(3.4). Then the problem*

$$x''(t) + p(t)f(t, x(t)) = r(t) \tag{3.8}$$

$$x(a) = r_1, \qquad x(b) = r_2, \qquad a < b; \;\; a, b \in \mathbb{R}^1; \;\; r_1, r_2 \in \overline{\mathbb{R}} \tag{3.9}$$

has exactly one solution x in $\mathcal{G}(\mathbb{R}^1)$.

THEOREM 3.3. *We assume the conditions (3.0)–(3.3) and (3.5). Then the problem (1.0); (1.2) has exactly one solution x in $\mathcal{G}(\mathbb{R}^1)$.*

REMARK 3.4. Let $f_1(t, u, v) = u, \;\; f_2(t, u, v) = 0$ and let $p \in L^1_{loc}(\mathbb{R}^1)$ (i.e. for every $K, \;\; p \in L^1(K)$). Moreover, let

$$\int_a^b |p|(t)dt < \frac{4}{b-a}. \tag{3.10}$$

Then f_1, f_2 and p have the properties (3.0)–(3.4) (see [11]).

REMARK 3.5. Let $\widetilde{\delta}$ be the generalized function defined by

$$R_{\widetilde{\delta}}(\phi, t) = \frac{\phi(-t)}{\int\limits_{-\infty}^{\infty} |\phi(-t)|dt}, \qquad \phi \in \mathcal{A}_1, \tag{3.11}$$

and let $f_1(t, u, v) = u, \quad f_2(t, u, v) = 0.$

Moreover, let $a = -1, \; b = 1$. Then $\widetilde{\delta}$ has the properties (3.1) and (3.4).

REMARK 3.6. Let $p, q \in L^1_{loc}(\mathbb{R}^1)$ and let $f_1(t, u, v) = u, \quad f_2(t, u, v) = v.$ Moreover, let

$$\int_a^b |p|(t)dt + \int_a^b |q|(t)dt < \frac{4}{b - a + 4}. \tag{3.12}$$

Then f_1, f_2, p and q have the properties (3.1)–(3.3) and (3.5) (see [12]).

4. Proofs

PROOF OF THEOREM 3.1. The proof of Theorem 3.1 is similar to that of Theorem 4.2 in [11]. We start from the problem

$$x''(t) + R_p(\phi, t) f_1(t, x(t), x'(t)) + R_q(\phi, t) f_2(t, x(t), x'(t)) = R_r(\phi, t), \quad \phi \in \mathcal{A}_1 \tag{4.1}$$

$$x(a) = R_{d_1}(\phi), \qquad x'(a) = R_{d_2}(\phi). \tag{4.2}$$

By (3.3) the problem (4.1), (4.2) has exactly one solution $x(\phi, t)$ in $\mathbb{R}^1$. We are going to prove $x(\phi, t) \in \mathcal{E}_M[\mathbb{R}^1]$. Indeed,

$$\begin{aligned} x(\phi_\varepsilon, t) = -\int_a^t (t - s)\,\big(&R_p(\phi_\varepsilon, s) f_1(s, x(\phi_\varepsilon, s), x'(\phi_\varepsilon, s)) \\ &+ (R_q(\phi_\varepsilon, s) f_2(s, x(\phi_\varepsilon, s), x'(\phi_\varepsilon, s)) - R_r(\phi_\varepsilon, s))\big)\, ds \\ &+ R_{d_1}(\phi_\varepsilon) + R_{d_2}(\phi_\varepsilon)(t - a). \end{aligned} \tag{4.3}$$

Using (3.0), (3.1), (3.3) and the Gronwall inequality we condude that there is $N \in \mathbb{N}$ such that: for all $\phi \in \mathcal{A}_N$ there are $c_0, \eta > 0$ such that

$$\|x(\phi_\varepsilon, t)\|^1_K \le c_0 \varepsilon^{-N} \qquad \text{if} \quad 0 < \varepsilon < \eta. \tag{4.4}$$

Hence, by (4.3) there is $N_r \in \mathbb{N}$ such that

$$\|D_r x(\phi_\varepsilon, t)\|_K \leq c_r \varepsilon^{-N_r} \tag{4.5}$$

for $\phi \in \mathcal{A}_{N_r}$ and $0 < \varepsilon < \eta_r$. Therefore $x(\phi, t) \in \mathcal{E}_M[\mathbb{R}^1]$.
Denoting by x the class of $x(\phi, t)$ in $\mathcal{G}(\mathbb{R}^1)$, we get that x is a solution of the problem (1.0), (1.1). Let $y \in \mathcal{G}(\mathbb{R}^1)$ be another solution of the problem (1.0), (1.1). Then

(4.6)
$$R_{y''}(\phi,t)+R_p(\phi,t)f_1(t, R_y(\phi,t), R_{y'}(\phi,t)) + R_q(\phi,t)f_2(t, R_y(\phi,t)R_{y'}(\phi,t))$$
$$=R_r(\phi,t) + R_n(\phi,t),$$

where $\phi \in \mathcal{A}_1$,

$$R_n(\phi,t) \in \mathcal{N}[\mathbb{R}^1] \tag{4.7}$$

$$R_{y(a)}(\phi) - R_{x(a)}(\phi) \in \mathcal{T}, \tag{4.8}$$

and

$$R_{y'(a)}(\phi) - R_{x'(a)}(\phi) \in \mathcal{T}. \tag{4.9}$$

In view of (3.1), (3.3), (4.3), the Gronwall inequality and (4.6)–(4.9) we deduce that (for $q \geq N_1'$ and $\phi \in \mathcal{A}_q$)

$$\|x(\phi_\varepsilon,t) - R_y(\phi_\varepsilon,t)\|_K^1 \leq \overline{c}\varepsilon^{\alpha(q)-N_1'} \quad \text{if} \quad 0 < \varepsilon < \overline{\eta}_0. \tag{4.10}$$

On the other hand, by (4.10), (4.3) and (4.6) we have

$$\|D_r(x(\phi_\varepsilon,t) - R_y(\phi_\varepsilon,t))\|_K \leq \overline{c}_r\varepsilon^{\alpha(q)-N_r'} \quad \text{for} \quad 0 < \varepsilon < \overline{\eta}_r. \tag{4.11}$$

This yields

$$x(\phi,t) - R_y(\phi,t) \in \mathcal{N}[\mathbb{R}^1] \tag{4.12}$$

and Theorem 3.1 is proved.

PROOF OF THEOREM 3.2. We consider the problem

$$x''(t) + R_p(\phi_\varepsilon,t)f_1(t,x(t)) = R_r(\phi_\varepsilon,t) \tag{4.13}$$

$$x(a) = R_r(\phi_\varepsilon), \qquad x(b) = R_{r_2}(\phi_\varepsilon), \qquad \phi \in \mathcal{A}_1, \quad t \in \mathbb{R}^1 \tag{4.14}$$

and the operation T_1 given by
(4.15)

$$T_1(y)(t) = -\int_a^b G(t,s)(R_p(\phi_\varepsilon, s)f_1(s, y(s)) - R_r(\phi_\varepsilon, s))ds + R_{r_1}(\phi_\varepsilon) + \frac{R_{r_2}(\phi_\varepsilon) - R_{r_1}(\phi_\varepsilon)}{b-a}(t-a),$$

where $y \in C_{[a,b]}$ and

$$G(t,s) = \begin{cases} \dfrac{(t-b)(s-a)}{b-a}, & \text{if} \quad a \le s \le t \le b \\ \dfrac{(a-t)(b-s)}{b-a}, & \text{if} \quad a \le t \le s \le b \end{cases} \tag{4.16}$$

Obviously, a function $x(\phi_\varepsilon, t) \in C^\infty[a,b]$ is a classical solution of the problem (4.13)–(4.14) (for a fixed $\phi_\varepsilon \in \mathcal{A}_1$) in the interval $[a,b]$ if and only if $x(\phi_\varepsilon, t)$ is a fixed point of the operation T_1. Taking into account that

$$\sup_{t,s \in [a,b]} |G(t,s)| = \frac{b-a}{4}, \tag{4.17}$$

we have

$$\|T_1(y) - T_1(z)\|_{[a,b]} \le I_1(p, \phi_\varepsilon)\left(\frac{b-a}{4}\right)\|y - z\|_{[a,b]}, \tag{4.18}$$

where $y, z \in C_{[a,b]}$. Applying the fixed point theorem of Banach we conclude that the problem (4.13)–(4.14) has exactly one solution $x(\phi_\varepsilon, t) \in C^\infty_{[a,b]}$ for small ε (see [4]). In view of (4.15) we deduce that for $\phi \in \mathcal{A}_N$ there are $c_0, \widetilde{c_0}, \widetilde{\eta_0} > 0$ such that

$$|x(\phi_\varepsilon, t_0)| \le c_0 \varepsilon^{-N} \tag{4.19}$$

and

$$|x'(\phi_\varepsilon, t_0)| \le \overline{c}_0 \varepsilon^{-N} \tag{4.20}$$

if $0 < \varepsilon < \widetilde{\eta_0}$ and $t_0 \in (a,b)$.
Thus

$$x(\phi, t_0), \quad x'(\phi, t_0) \in \mathcal{E}_M. \tag{4.21}$$

Let $\overline{x}(\phi_\varepsilon, t)$ be a solution of the problem

$$x'' + R_p(\phi_\varepsilon, t) f_1(t, x(t)) = R_r(\phi_\varepsilon, t) \tag{4.22}$$

$$x(t_0) = x(\phi_\varepsilon, t_0), \qquad x'(t_0) = x'(\phi_\varepsilon, t_0) \tag{4.23}$$

for $t \in \mathbb{R}^1$ and small ε. Then

$$\overline{x}(\phi_\varepsilon, t) = x(\phi_\varepsilon, t) \qquad \text{for} \quad t \in [a, b]. \tag{4.24}$$

and by Theorem 3.1

$$x(\phi, t) \in \mathcal{E}_M[\mathbb{R}^1].$$

If we define x as the class of $x(\phi, t)$ in $\mathcal{G}(\mathbb{R}^1)$, then x is a solution of the problem (3.8)–(3.9).

To prove uniqueness of solutions of the problem (3.8)–(3.9) we observe that if $y \in \mathcal{G}(\mathbb{R}^1)$ is another solution of the problem (3.8)–(3.9), then

$$R_{y''}(\phi, t) + R_p(\phi, t) f_1(t, R_y(\phi, t)) = R_r(\phi, t) + R_n(\phi, t), \tag{4.25}$$

where $\phi \in \mathcal{A}_1$,

$$R_n(\phi, t) \in \mathcal{N}[\mathbb{R}^1], \tag{4.26}$$

$$R_{y(a)}(\phi) - R_{x(a)}(\phi) \in \mathcal{T} \tag{4.27}$$

and

$$R_{y(b)}(\phi) - R_{x(b)}(\phi) \in \mathcal{T}. \tag{4.28}$$

Relations (4.13)–(4.15) and (4.25)–(4.28) yield for $q \geq N_1$ and $\phi \in \mathcal{A}_q$

$$\begin{aligned} &\|x(\phi_\varepsilon, t) - R_y(\phi_\varepsilon, t)\|_{[a,b]} \leq c\varepsilon^{\alpha(q) - N_1} \\ &+ I_1(p, \phi_\varepsilon) \left(\frac{b-a}{4}\right) \|x(\phi_\varepsilon, t) - R_y(\phi_\varepsilon, t)\|_{[a,b]} \quad \text{if} \quad 0 < \varepsilon < \eta_1. \end{aligned} \tag{4.29}$$

Therefore

$$\|x(\phi_\varepsilon, t) - R_y(\phi_\varepsilon, t)\|_{[a,b]} \leq \widetilde{c}\varepsilon^{\alpha(q) - N_1} \tag{4.30}$$

for small ε and $\phi \in \mathcal{A}_q$.

Similarly

$$\|x'(\phi_\varepsilon, t) - R_{y'}(\phi_\varepsilon, t)\|_{[a,b]} \le \widetilde{c_1}\varepsilon^{\alpha(q)-N_2} \tag{4.31}$$

for $0 < \varepsilon < \eta_2$ and $\phi \in \mathcal{A}_q$, where $q \ge N_2$.
This yields

$$R_x(\phi, t) - R_y(\phi, t) \in \mathcal{N}[\mathbb{R}^1] \tag{4.32}$$

and

$$x'(\phi, t) - R_{y'}(\phi, t) \in \mathcal{N}[\mathbb{R}^1] \tag{4.33}$$

for every $t \in (a, b)$.
Using Theorem 3.1 we infer that

$$x = y. \tag{4.34}$$

This proves the theorem.

PROOF OF THEOREM 3.3. The proof of Theorem 3.3 is similar to the proof of Theorem 3.2. To this purpose we examine the problem

$$x'' + R_p(\phi_\varepsilon, t)f_1(t, x(t), x'(t)) + R_q(\phi_\varepsilon, t)f_2(t, x(t), x'(t)) = R_r(\phi_\varepsilon, t), \tag{4.35}$$

$$x(a) = R_{r_1}(\phi_\varepsilon), \qquad x(b) = R_{r_2}(\phi_\varepsilon), \qquad \phi \in \mathcal{A}_1, \quad t \in \mathbb{R}^1 \tag{4.36}$$

and the operation T_2:

$$\begin{aligned} T_2(y)(t) = -\int_a^b & G(t,s)(R_p(\phi_\varepsilon, s)f_1(s, y(s), y'(s)) \\ & + R_q(\phi_\varepsilon, s)f_2(s, y(s), y'(s)) - R_r(\phi_\varepsilon, s))ds \\ & + R_{r_1}(\phi_\varepsilon) + \frac{R_{r_2}(\phi_\varepsilon) - R_{r_1}(\phi_\varepsilon)}{b-a}(t-a), \end{aligned} \tag{4.37}$$

where $y \in C^1[a, b]$. Then

$$\|T_2(y) - T_2(z)\|^1_{[a,b]} \le \left(\frac{b-a+4}{4}\right) I_2(p, q, \phi_\varepsilon)\|y - z\|^1_{[a,b]}, \tag{4.38}$$

where $y, z \in C^1_{[a,b]}$. Hence we deduce that the problem (4.35)–(4.36) has exactly one solution $x(\phi_\varepsilon, t)$ for $t \in \mathbb{R}^1$, $\phi \in \mathcal{A}_1$ and small ε. We observe that $x(\phi, t) \in \mathcal{E}_M[\mathbb{R}^1]$. If $y \in \mathcal{G}(\mathbb{R}^1)$ is another solution of the problem (1.0); (1.2), then

$$\begin{aligned} &\|x(\phi_\varepsilon, t) - R_y(\phi_\varepsilon, t)\|^1_{[a,b]} \\ (4.39) \qquad &\leq \left(\frac{b-a+4}{4}\right) I_2(p, q, \phi_\varepsilon)\|x(\phi_\varepsilon, t) - R_y(\phi_\varepsilon, t)\|^1_{[a,b]} \\ &+ c_1\varepsilon^{\alpha(q)-N_1} \text{ if } 0 < \varepsilon < \eta_1 \quad (\phi \in \mathcal{A}_q \text{ for } q \geq N_1). \end{aligned}$$

Thus, by virtue of (4.39), we obtain

$$(4.40) \qquad \|x(\phi_\varepsilon, t) - R_y(\phi_\varepsilon, t)\|^1_{[a,b]} \leq \tilde{c}_1\varepsilon^{\alpha(q)-N_1} \qquad \text{if} \quad 0 < \varepsilon < \eta_1.$$

Consequently,

$$(4.41) \qquad x(\phi, t) - R_y(\phi, t) \in \mathcal{N}[\mathbb{R}^1].$$

which completes the proof of Theorem 3.3.

5. Final remarks

REMARK 5.1. If $G_1, G_2 \in C^\infty(\mathbb{R}^1)$, then the choice of the representatives $R_i(\phi, t) = G_i(t)$ $(i = 1, 2)$ shows that definition of the superposition gives back the classical C^∞ function $f(t, G_1, G_2)$ (if $f \in O_M(K, \mathbb{R}^2)$). In case the functions G_i are only continous functions it has already been ascertained that the above coherence results does not hold even for multiplication.

EXAMPLE 5.1. Let G_1, G_2 be continous functions defined by

$$(5.0) \qquad G_1(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ t, & \text{if } t > 0, \end{cases}$$

$$(5.1) \qquad G_2(t) = \begin{cases} t, & \text{if } t \leq 0, \\ 0, & \text{if } t > 0. \end{cases}$$

Then their classical product in $C(\mathbb{R}^1)$ is 0. Their product in $\mathcal{G}(\mathbb{R}^1)$ is the class of

$$(5.2) \qquad R(\phi, t) = \int_{-\infty}^{\infty} G_1(t+u)\phi(u)du \cdot \int_{-\infty}^{\infty} G_2(t+u)\phi(u)du,$$

where $\phi \in \mathcal{A}_1$. By [2] (p. 16) we have

$$R(\phi, t) \notin \mathcal{N}[\mathbb{R}^1]. \tag{5.3}$$

REMARK 5.2. We denote the product in $\mathcal{G}(\mathbb{R}^1)$ by $\odot$ to avoid confusion with the classical product. Now, we consider the equations

$$x''(t) = G_1(t)x'(t) + G_2'(t), \tag{5.4}$$

$$x''(t) = G_1(t) \odot x'(t) + G_2'(t), \tag{5.5}$$

where G_1 and G_2 are defined by (5.0)–(5.1). Let

$$\widetilde{G}_2(t) = \int_0^t G_2(s)ds. \tag{5.6}$$

Then $x = \widetilde{G}_2$ is a classical solution of the equation (5.4) (in the Carathéodory sense). On the other hand $x = \widetilde{G}_2$ is not a solution of the equation (5.5) in the Colombeau algebra $\mathcal{G}(\mathbb{R}^1)$ (because $G_1 \odot G_2$ is not zero in $\mathcal{G}(\mathbb{R}^1)$).

REMARK 5.3. It is known that every distribution is moderate (see [2]). On the other hand, L. Schwartz proves in [17] that there does not exist an algebra A such that the algebra $C(\mathbb{R}^1)$ of continuous functions on $\mathbb{R}^1$ is subalgebra of A, the function 1 is the unit element in A, elements of A are "C^∞" with respect to a derivation which coincides with usual one in $C^1(\mathbb{R}^1)$, and such that the usual formula for the derivation of a product holds. As consequence multiplication in $\mathcal{G}(\mathbb{R}^1)$ does not coincide with usual multiplication of continuous functions.

To repair the consistency problem for multiplication (and superposition) we give the definition introduced by J. F. Colombeau in [2].

An element u of $\mathcal{G}(\mathbb{R}^1)$ is said to admit a member $w \in \mathcal{D}'(\mathbb{R}^1)$ as the associated distribution, if it has a representative $R_u(\phi, t)$ with the following property: for every $\psi \in \mathcal{D}(\mathbb{R}^1)$ there is $N \in \mathbb{N}$ such that for every $\phi \in \mathcal{A}_N$ we have

$$\int_{-\infty}^{\infty} R_n(\phi_\varepsilon, t)\psi(t)dt \to w(\psi) \qquad \text{as} \quad \varepsilon \to 0. \tag{5.7}$$

COROLLARY 5.1. *We assume*

(5.8) $p, q, r \in L^1_{loc}(\mathbb{R}^1)$,
(5.9) f_1, f_2 *have the properties (3.2)–(3.3),*
(5.10) $d_1, d_2 \in \mathbb{R}^1$,
(5.11) $x \in \mathcal{G}(\mathbb{R}^1)$ *is the solution of the problem (1.0)–(1.1) ,*
(5.12) $\widetilde{x}$ *is the solution of the problem (1.0)–(1.1) in the Caratheodory sense.*

Then x admits an associated distribution which equals $\widetilde{x}$.

This follows from the fact that $p * \phi_\varepsilon \to p$, $q * \phi_\varepsilon \to q$ and $r * \phi_\varepsilon \to r$ in $L^1_{loc}(\mathbb{R}^1)$ (see [1]) and the continuous dependence of $\widetilde{x}$ on coefficients p, q and r.

Using arguments similar to these in Corollary 5.1, we get

COROLLARY 5.2. *We assume*

(5.13) $p, q, r \in L^1_{loc}(\mathbb{R}^1)$,
(5.14) p, q *satisfy (3.5),*
(5.15) f_1, f_2 *have the properties (3.2)–(3.3),*
(5.16) $x \in \mathcal{G}(\mathbb{R}^1)$ *is the solution of the problem (1.0); (1.2),*
(5.17) $\widetilde{x}$ *is the solution of the problem (1.0); (1.2) in the Caratheodory sense.*

Then x admits an associated distribution which equals $\widetilde{x}$.

COROLLARY 5.3. *We assume*

(5.18) $p, r \in L^1_{loc}(\mathbb{R}^1)$,
(5.19) p *satisfies (3.4),*
(5.20) f_1 *has the property (3.2)–(3.3),*
(5.21) $x \in \mathcal{G}(\mathbb{R}^1)$ *is the solution of the problem (3.8)–(3.9),*
(5.22) $\widetilde{x}$ *is the solution of the problem (3.8)–(3.9) in the Caratheodory sense.*

Then x admits an associated distribution which equals $\widetilde{x}$.

If $p \in C^\infty(\mathbb{R}^1)$, then $p(t) - \int_{-\infty}^{\infty} p(t+u)\phi(u)du \in \mathcal{N}[\mathbb{R}^1]$, where $\phi \in \mathcal{A}_1$ (see[2]). Hence, we get

COROLLARY 5.4. *We assume*

(5.23) $p, q, r \in C^\infty(\mathbb{R}^1)$,
(5.24) f_1, f_2 *have the properties (3.2)–(3.3),*
(5.25) $d_1, d_2 \in \mathbb{R}^1$.

Then the classical and the generalized solution (i.e. solution in the Colombeau algebra) of the problem (1.0)–(1.1) give rise to the same elements of $\mathcal{G}(\mathbb{R}^1)$.

COROLLARY 5.5. *We assume*

(5.26) $p \in C^\infty(\mathbb{R}^1)$,
(5.27) f_1 *has the properties (3.2)–(3.3),*
(5.28) p *has the property (3.4),*
(5.29) $r_1, r_2 \in \mathbb{R}^1$.

Then the classical and the generalized solution of the problem (3.8)–(3.9) give rise to the same elements of $\mathcal{G}(\mathbb{R}^1)$.

COROLLARY 5.6. *We assume*

(5.30) $p, q, r \in C^\infty(\mathbb{R}^1)$,
(5.31) f_1, f_2 *have the properties (3.2)–(3.3),*
(5.32) p, q *have the property (3.5),*
(5.33) $r_1, r_2 \in \mathbb{R}^1$.

Then the classical and the generalized solution of the problem (1.0); (1.2) give rise to the same elements of $\mathcal{G}(\mathbb{R}^1)$.

REMARK 5.4. Non continuous solutions of ordinary differential equations can be considered in an other way (for example [3], [5]–[11], [13]–[16] and [18].

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Uniwersytet Śląski
Instytut Matematyki
ul. Bankowa 14
40–007 Katowice